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LETTER TO THE EDITOR

Representation-free evaluation of the eigenvalues of the class-sums of the symmetric group

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Abstract. Explicit expressions for the eigenvalues of the single-cycle class-sums $[(p)(1)^{n-p}]_n$ of the symmetric group (S_n) are constructed in a manner which makes no use of representation theoretic data. The expressions obtained consist of polynomials in the symmetric power sums over the 'contents' of the Young diagram specifying the irreducible representation. The construction uses a conjecture concerning the structure of these expressions and a lemma concerning the vanishing of these expressions when evaluated for $n < p$.

The sum of the elements of the symmetric group S_n which consists of a cycle of length p and $n-p$ cycles of unit length is an element of the corresponding group algebra which we denote by the symbol $[(p)(1)^{n-p}]_n$, or, for brevity, by $[(p)]_n$. The centre of the symmetric group algebra is generated by means of polynomials over the single-cycle class-sums $[(1)^n]_n, [(2)]_n, \ldots, [(n)]_n$ [1]. The eigenvalues of these class-sums are closely related to the characters of the irreducible representations (irreps) of the symmetric group [2]. They can be labelled by means of partitions of n, which are commonly represented graphically by Young diagrams. Denoting each box in the Young diagram by a row index i (running from top to bottom) and a column index j (running from left to right) we refer to the difference $j-i$ as the content of the box (i, j) [3].

We shall use the power sums over the contents of a Young diagram Γ, which are defined by

$$\sigma_r = \sum_{(i,j)\in\Gamma} (j-i)^r \qquad r = 1, 2, \ldots.$$

The eigenvalue of the class-sum $[(p)]_n$ corresponding to the irrep Γ, which is associated with the partition $[\lambda_1, \lambda_2, \ldots, \lambda_k]$ $(\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$, where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0)$ is given by an expression which was presented by Macdonald [3], following Frobenius. This expression involves the n strictly decreasing variables

$$\mu_i = \lambda_i - i + n \qquad i = 1, 2, \ldots, n.$$

It can be written in the form

$$\lambda^{\Gamma}_{[(p)]_n} = \frac{1}{p}\sum_{i=1}^{n} \mu_i(\mu_i - 1)\cdots(\mu_i - p + 1)\prod_{j\neq i}\frac{(\mu_i - \mu_j - p)}{(\mu_i - \mu_j)}. \qquad (1)$$

Equation (1) was recently used to show that these same eigenvalues can also be written as polynomials in the $p-1$ power sums $\sigma_1, \sigma_2, \ldots, \sigma_{p-1}$, with coefficients which are polynomials in n [4]. The actual evaluation of the coefficients, as presented in [4] and [5], was rather heuristic. For the class-sum $[(p)]_n$ one had to use the characters corresponding to the irreps of the symmetric groups $S_p, S_{p+1}, \ldots, S_{p+k}$, where k is the maximal order in n of the expression sought. Note that even subject to this procedure one obtains an impressive extrapolation since the expression obtained for any particular value of p is valid for arbitrary n. Some further advantages and applications of this expression for the eigenvalues were discussed in [4].

An alternative construction for the expressions for the eigenvalues of the single-cycle class-sums as polynomials over the symmetric power-sums in the contents of Young diagrams will now be presented. It is based on a statement concerning the structure of these polynomials (conjecture 1) and on a lemma which (along with conjecture 2) enables the formulation of a system of linear equations for the coefficients in each one of these polynomials. While the proof of the lemma uses equation (1) and the proofs of the conjectures remain to be established, the actual construction of these polynomials does not involve the use of any representation-theoretic data.

Conjecture 1. The expression for $\lambda^{\Gamma}_{[(p)]_n}$ in terms of $\sigma_1, \sigma_2, \ldots, \sigma_{p-1}$ is obtained as follows. Each partition of $p+1$ into parts which are not smaller than 2, i.e.

$$2n_2 + 3n_3 + \cdots + (p+1)n_{p+1} = p + 1$$

gives rise to a term of the form $f_{n_2}(n)\prod_{i=1}^{p-1}\sigma_i^{n_{i+2}}$. $f_{n_2}(n)$ is a polynomial of order n_2 in n, whose coefficients depend on the partition of $p+1$ to which it corresponds.

Example 1. $p = 2$. The only relevant partition of three is $\{n_3 = 1\}$, hence $\lambda^{\Gamma}_{[(2)]_n} = \alpha\sigma_1$, where α is a constant which has yet to be determined.
$p = 3$. Four can be partitioned in two ways: $\{n_4 = 1\}$ and $\{n_2 = 2\}$. Hence, $\lambda^{\Gamma}_{[(3)]_n} = \alpha\sigma_2 + f_2(n)$.
$p = 4$. The relevant partitions of five are $\{n_5 = 1\}$ and $\{n_3 = 1, n_2 = 1\}$, resulting in $\lambda^{\Gamma}_{[(4)]_n} = \alpha\sigma_3 + f_1(n)\sigma_1$.

The conjecture has been verified, using the data in [5], for all $p \leqslant 14$.

Lemma. For $p > n$, $\lambda^{\Gamma}_{[(p)]_n}$ vanishes.

Proof. We shall show that each one of the summands in equation (1) vanishes. Consider the ith summand and let $\lambda_i = k$. It follows that $\lambda_1 + \lambda_2 + \cdots + \lambda_i \geqslant ki$ and $\lambda_j = 0$ for $n + 1 - i(k-1) \leqslant j \leqslant n$. We consider two separate cases:

(i) $\mu_i \leqslant p - 1$. Since $\mu_i \geqslant 0$ it follows that one of the factors $\mu_i, \mu_i - 1, \ldots, \mu_i - p + 1$ vanishes. This case contains the rows for which $\lambda_i = 0$.

(ii) $\mu_i \geqslant p > n$. For $n + 1 - i(k-1) \leqslant j \leqslant n$ $\mu_i - \mu_j - p$ increases by steps of 1 from $n + 1 - p - (i-1)k \leqslant 0$ to $\mu_i - p \geqslant 0$. Hence, for some j in this range the factor $\mu_i - \mu_j - p$ vanishes.

This concludes the proof of the lemma. □

Since the system of linear equations obtained for the coefficients of the various terms in $\lambda^{\Gamma}_{[(p)]_n}$ is homogeneous, it only determines the coefficients up to a common multiplicative factor. To overcome this difficulty we state, on the basis of the evidence available in [5] for $2 \leqslant p \leqslant 14$,

Conjecture 2. The coefficient of σ_{p-1} in the expression for $\lambda^{\Gamma}_{[(p)]_n}$ is equal to 1.

Remark 1. For $n = 0$ and $n = 1$ $\sigma_i = 0$, for all i.

Remark 2. For odd p the partition of $p+1$ into $n_2 = (p+1)/2$ corresponds to a polynomial, which by lemma 1 has to vanish for $n = 0$ and $n = 1$. Thus, this polynomial is of the form $n(n-1)f_{(p-3)/2}(n)$.

Example 2. By conjecture 2, $\alpha = 1$ in all the cases discussed in example 1. For $p = 2$ the expression for the eigenvalues corresponding to arbitrary irreps is completely determined. The result is in agreement with Jucys [6] and Suzuki [7].

For $p = 3$ we obtain $f_2(n)$ as follows. By remark 2 $f_2(n) = \beta n(n-1)$. To determine β we consider the irrep [2] of S_2: since in this irrep $\sigma_2 = 1$, we obtain $\lambda^{[2]}_{[(3)]_2} = 1 + 2\beta = 0$, i.e. $\beta = -\frac{1}{2}$. Consequently, $\lambda^{\Gamma}_{[(3)]_n} = \sigma_2 - \frac{1}{2}n(n-1)$, in agreement with Jucys [6] and Suzuki [7].

For $p = 4$ we consider the irreps [2] and [3] of S_2 and S_3, respectively. We note that $\sigma_1 = \sigma_3 = 1$ for the former and $\sigma_1 = 3$, $\sigma_3 = 9$ for the latter. Hence, $f_1(2) = -1$, $f_1(3) = -3$, i.e. $f_1(n) = 3 - 2n$ or $\lambda^{\Gamma}_{[(4)]_n} = \sigma_3 - (2n-3)\sigma_1$ in agreement with [4] and [5].

The actual application of the algorithm presently discussed to the evaluation of the eigenvalues of single-cycle class-sums with cycle lengths up to 20 will be presented elsewhere [8]. For a discussion of quantum-mechanical applications we refer to [1, 4, 5]. It is hoped that some of the readers will be interested in proving the two conjectures presented above.

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